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Effect of Crew Motions on Spacecraft Orientation

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Crew motions can affect the angular orientation of a spacecraft. Such motions are somewhat random and must be treated stochastically. Linearized equations of motion for the spacecraft are solved to obtain a simple relationship between individual crew motions and the angular change in spacecraft orientation. The additive effect of many motions is found by treating the problem as a Markovian random walk. Starting from a known distribution of crew motions, the mean waiting time required to reach predetermined maximum allowable angular deviations is found; the inverse of the waiting time can then be used to determine the frequency of control jet firing and fuel consumption rate. For heavy spacecraft, where many individual motions occur between jet firings, the diffusion equation applies. For smaller spacecraft, motions are treated as discrete steps using matrix equations.

Nomenclature

a = variance of the angular rate of change
B = number of subperiods of length h
b = mean angular rate of change
$E(\ )$ = expectation
H = angular momentum
h = length of subperiods
I = inertia
I = unit matrix
K = reduced mass
M = mass of spacecraft
m = internal moving point mass
P = probability that an event will occur
$\tilde{P}$ = transition matrix
p = Laplace transform parameter (angle)
$\tilde{p}$ = one-step transition matrix
$\hat{p}$ = modified one-step transition matrix (reflecting boundaries)
$\bar{r}_1$ = position vector of mass m in X, Y, Z coordinates
$\bar{r}_2$ = position vector of differential mass dM in X, Y, Z coordinates
S = surface bounded by the path of m
s = Laplace transform parameter (time)
T,t,τ = time
u = probability density function for X (angle)
Var = variance
W = waiting time
X = angle of rotation
δ_{ij} = Kronecker delta
η = to be defined by Eq. (58)
λ = number of moves/sec
$\tilde{\lambda}$ = matrix of λ's
μ = mean number of moves, $\mu = \lambda T$
ν = probability vector
$\bar{\rho}$ = position vector of mass m in (1), (2), (3) coordinate system
$\bar{\rho}_M$ = position vector of differential mass dM in (1), (2), (3) coordinate system
φ = angle of rotation
ω = angular velocity

Subscripts

i,j,k,l,N = indices
b = upper boundary for diffusion analysis
c = center of mass
0 = nominal position; or boundary value of λ
r = relative to moving coordinates
γ = upper boundary state for discrete-step analysis

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Introduction

SINCE space flight has become a practical accomplishment there has been an increased interest in the motion of bodies in a reduced force, or zero force environment. This is because, at large distances from the earth, gravitational-gradient torques and atmospheric drag have a diminishing influence on spacecraft motion. There, the major spacecraft torques originate internally and are caused by the crew and machinery. To remove the effects of these torques, attitude stabilization will be needed. Meirovitch[1] considered spin stabilization for unmanned spacecraft. However, a manned mission would require a more complex mechanism, such as control jets or reaction wheels; either mechanism expends valuable energy. A method is needed to predict the required amount of energy for manned missions.

Pistiner[2] developed techniques to analyze on-off control systems with reaction jets; but he did not consider the effect of disturbing torques on the vehicle. Regetz and Nelson[3] considered a similar system and included a disturbance torque; but they considered only a constant torque. To define an experiment for an Apollo mission, Tewell and Murrish[4] considered torques caused by crew motion. They studied the actual motions of a man, and measured the associated forces and moments. Although they considered varying torques, their approach was deterministic. The present paper uses such torques as disturbances, but treats the torques and resultant spacecraft motions stochastically. Tewell and Murrish's[4] studies of crew disturbances were synthesized into an analytic representation of disturbances by Hendricks and Johnson.[5]

Because a crew member cannot be expected to use exactly the same sequence of motions each time he performs a specific task, the calculation of the spacecraft angular drift rate seems to have an aura of unpredictability. However, one would expect crew motions to be statistically stationary, at least. Then, frequency-vs-magnitude histograms of disturbances for specific and/or varied tasks may be used to predict crew-induced changes in spacecraft attitude. These histograms are the basic starting data for the present analysis. Fundamental to their determination is the work of Tewell and Murrish,[4] and Hendricks and Johnson.[5]

In the present paper, an on-off control system is considered where random disturbing torques act upon the spacecraft. The "waiting time" is determined. This is the time required for the spacecraft to move from a nominal orientation to preselected limits on the angular deviation. The inverse of the mean waiting time is the mean frequency of firing of the control jets; either determines the mean rate of energy consumption. Two methods are derived to predict the mean waiting time. Each method pertains to a very slowly rotating spacecraft with no external moments acting upon it; the spacecraft responds only to random (but statistically stationary) internal disturbances. This is a problem in rotational dynamics where the total angular momentum of the spacecraft system (spacecraft plus contents) is always zero. The solutions are obtained in two steps: first, the spacecraft angular deviation due to a single internal motion is determined (deterministically). Then, through a histogram of either internal motions or spacecraft responses, the additive effect of many motions is considered (stochastically).

The relationship between dynamics and the equations from probability theory is shown, and the equations solved for the aforementioned problem. The derivation of equations for the mean and variance of the waiting time (from the Kolmogorov equation solution for location probabilities) is conducted in the transformed plane. An iterative method of obtaining the stationary solution of an ergodic matrix is developed.

Table 1 Typical Values for the Dynamic Properties of Apollo Applications Spacecraft[a]

Configuration	Mass, kg	Inertia, kg-m²		
		I_{11}	I_{22}	I_{33}
1	11 300	18 800	63 500	64 800
2	31 200	160 000	2 668 000	2 620 000
3	39 800	423 000	2 954 000	2 690 000

[a] From Tewell and Murrish.[4]

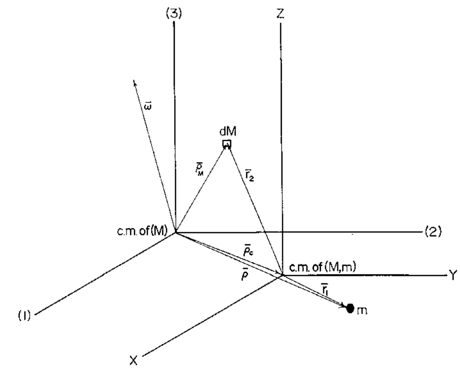

Fig. 1 Sketch of the coordinate systems. Axes X, Y, Z, are inertial. Axes (1), (2), (3), are along the principal axes of the spacecraft, and rotate with it. For clarity, $\bar{\rho}$, $\bar{\rho}_c$, and $\bar{r}_1$ are drawn not parallel, although they actually are so.

Relationship between Internal Movement and Spacecraft Reorientation

Typically, the principal moments of inertia of a spacecraft will be much larger than the corresponding quantities for its crew. Spacecraft data for three Apollo applications configurations are listed in Table 1. A man moving his entire body would provide a moving mass of about 100 kilograms and, inasmuch as he would normally be inside the spacecraft, the moments of inertia associated with him would be much less than the principal moments of inertia of the spacecraft. These points will be the basis for subsequent approximations.

In addition, the analysis is greatly simplified by restricting the spacecraft maximum angular deviation from the nominal position to $\pm 5°$ so that the direction cosines of the angles of deviation remain near unity. Doolin[6] lists some typical attitude requirements. A solar power array, which is a paraboloidal reflector used to heat a boiler for a heat engine, required an alinement accuracy of 0.1 sec of arc be maintained. And an orbital telescope would require 0.01 sec. But most mission specifications are not so stringent, and can be expected to require an accuracy of from 0.1° to a few degrees misalinement.

The first step is to relate the rotation of the spacecraft (of total mass M) to the internal motions of the point mass, m. (The effect of motion of internal masses with significant rotational inertia may, if necessary, be obtained later by changing m to dm and integrating.) The momentum of the M,m system (combined or total system) about the center of mass of the M,m system must remain constant because no external torques are applied. Fig. 1 shows the coordinate systems used; X, Y, Z coordinates are inertial (nonrotating, nonaccelerating) coordinates taken at the center of mass of the M,m system; the (1), (2), (3) coordinates are taken from the center of mass of the spacecraft itself, are alined with the principal moments of inertia of the spacecraft, and rotate with the spacecraft. The axes of the two coordinate systems are initially parallel. The vectors $\bar{r}_1$ and $\bar{r}_2$ are from the center of mass (c.m.) of the M,m system to the masses m and dM (a spacecraft mass element), respectively. The vectors $\bar{\rho}$ and $\bar{\rho}_M$ are from the c.m. of the M system to m and dM, respectively, and $\bar{\rho}_c$ is the distance of the c.m. of the M,m system from the c.m. of the M system. For constant angular momentum in the X, Y, Z coordinate system

$$\bar{H}_c = 0 = \bar{r}_1 \times m(d\bar{r}_1/dt) + \int_M \bar{r}_2 \times (d\bar{r}_2/dt)dM = (\bar{\rho} - \bar{\rho}_c) \times (d/dt)m\,(\bar{\rho} - \bar{\rho}_c) + \int_M (\bar{\rho}_M - \bar{\rho}_c) \times (d/dt)(\bar{\rho}_M - \bar{\rho}_c)\,dM \quad (1)$$

Because the (1), (2), (3) system rotates at angular velocity $\bar{\omega}$ and has its origin at the spacecraft c.m., Eq. (1) leads to

$$[\bar{\bar{I}} + K(\bar{\bar{U}}\rho^2 - \bar{\rho}\bar{\rho})]\cdot\bar{\omega} + K\bar{\rho} \times (d\bar{\rho}/dt)_r = 0 \quad (2)$$

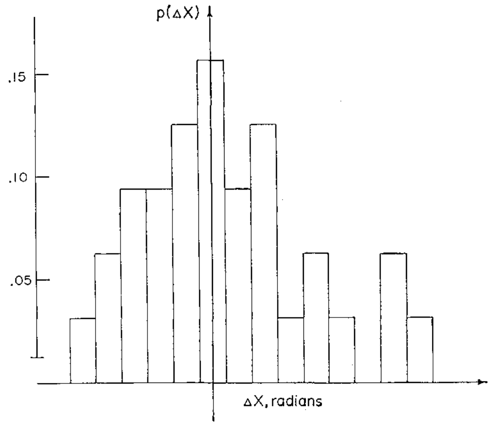

Fig. 2 Histogram showing the probability that ΔX will have a given magnitude. The abscissa for the histogram for ΔX is linearly related to the abscissa for the histogram of $[Mm/(M + m)s_i]$. The ordinates are identical. One such histogram for ΔX must be obtained experimentally for each of the three axes. Similar histograms could be constructed from data like that described in Refs. 4 and 5.

where $\bar{\bar{I}}$ is the inertia dyadic of the spacecraft, $\bar{\bar{U}}$ is the unit dyadic, the subscript r denotes a velocity measured relative to the (1), (2), (3) coordinate system, and

$$K = Mm/(M + m) \tag{3}$$

For crews inside the spacecraft $K(\bar{\bar{U}}\rho^2 - \bar{\rho}\bar{\rho}) \ll \bar{\bar{I}}$ and, consequently, by neglecting $K(\bar{\bar{U}}\rho^2 - \bar{\rho}\bar{\rho})$ Eq. (2) can be written as

$$-\bar{\bar{I}}\cdot\bar{\omega} = K\bar{\rho} \times (d\bar{\rho}/dt)_r \tag{4}$$

Suppose a crew member moves a hand from the armrest of his chair, operates a switch or valve on the instrument console, then returns his arm to the armrest. All point masses, m (of the arm), describe a closed path in space. Such motion is defined as "complete" motion; as will be discussed later, all motions may be considered to be of this type. During the motion, the tip of the vector $\bar{\rho}$ follows a point mass m and if m describes a closed path in the time period $(t_2 - t_1)$, the effect of the complete motion is determined by the integration of Eq. (4)

$$-\int_{t_1}^{t_2}\bar{\bar{I}}\cdot\bar{\omega}dt = K\oint_{t_1}^{t_2}\bar{\rho} \times (d\bar{\rho}/dt)_r\, dt \tag{5}$$

With the restriction that the total angular deviation of the spacecraft is small, Eq. (5) may be integrated (using Stokes' theorem) to yield

$$-\bar{\bar{I}}\,\Delta\bar{\varphi} = 2K \iint_S \bar{n}\, dS \tag{6}$$

where $\bar{n}$ is the unit normal to a surface element dS; the surface S is bounded by the path of m. The components of $\bar{\varphi}$ are proportional to the projections of S on the coordinate planes

$$-\Delta\varphi_N = 2KS_N/I_{NN} \tag{7}$$

where $\Delta\varphi$ is the change in angle between the inertial coordinates and spacecraft coordinates and, for example, S_1 is the projection of S on the (2), (3) coordinate plane. Equation (7) shows that the angular changes among the three coordinates are uncoupled when the motions are limited to small angular deviations and the spacecraft moments of inertia are much larger, respectively, than the effective moments of inertia of the moving mass.

Stochastic Analysis

When the sums of small rotations about each of the axes are themselves small, a motion about a coordinate axis is independent of motions about the other axis. Henceforth, the remaining equations will be derived for the rotational component about one coordinate axis.

In the most usual situation, one complete motion will not result in the spacecraft rotating enough to reach the prechosen limits to its angular deviation. The accumulated effect of several motions must be considered. The exact pattern of motion that a crew member might make is not completely predictable; there will be variations among repetitions of even a simple task, such as the operation of a switch. However, experiments can be devised (Tewell and Murrish)[4] to measure the frequency of occurrence of given magnitudes of either KS_N or φ_N for short, simple tasks, or for long-term, complex tasks. The crew member should not be restricted artificially in his movements during these experiments, for the data should represent motions made during actual spacecraft operation. Experimentally, the effects of motions are sometimes recorded as reactive forces on a platform, from which torques can be calculated. A "complete" motion can be defined here as the points between where the velocity is other than zero (start), and where the velocity is next zero (end). Several such motions may occur before the moving mass actually completes a closed path, but each (partial) motion induces a smaller rotation, while the rate at which motions (appear to) occur increases in compensation.

Typical frequencies of occurrence of motions of a given magnitude about one coordinate axis are sketched in Fig. 2. Figure 2 does not depict data from an actual experiment. The values were deliberately chosen so that the final distribution is not quite normal; the stochastic analysis is not restricted to normal distributions.

To circumvent a notational difficulty, a new notation is introduced in Fig. 2. Let X represent the total angular displacement between a corresponding pair of intertial spacecraft axes. Then let X_1 represent X at time t_1 (X represents the magnitude of the component of φ being considered).

The total angular displacement at time t_n is

$$X_n = \Delta X_1 + \Delta X_2 \ldots \Delta X_n \tag{8}$$

where

$$\Delta X_i = X_i - X_{i-1} \tag{9}$$

The value of ΔX_i, given that a motion occurs, is specified by Fig. 2. With one exception, the value of ΔX_i is assumed independent of X_i, so that Fig. 2 applies for each step. The exception occurs when the step is such that a preselected boundary is reached. Here, truncation occurs; the mathematical means for handling these boundary conditions will be developed in later sections.

The qualifying statement "given that a motion occurs" implies a probability distribution for the number of motions made during a given period. The total angular deviation depends upon both the size and the number of steps which occurred.

Suppose that the histogram of Fig. 2 was obtained by observing the actions (or effect of the actions) of a crew member over some period of T. The number of motions made during the period was counted. Divide T into B intervals of length $h = T/B$. Let the probability that one motion is made during an interval be

$$P[n = 1] = \lambda h$$

where λ is the average rate at which motions are made, and $h \ll 1/\lambda$.

It follows that the probability that the number of motions is "n" in the interval T is given by the Bernoulli probability law

$$P[n|B \text{ trials during period } T] = [B!/(B - n)!n!](\lambda h)^n(1 - \lambda h)^{B-n} \tag{10}$$

and that, as the interval h decreases, this probability is given by the Poisson distribution

$$p[n] = (\mu^n e^{-\mu}/n!) \quad (11)$$

where

$$\lim_{\substack{B\to\infty \\ h\to 0}} B\lambda h = \mu = \lambda T \quad (12)$$

Equation (11) implies that $1/\lambda$ is the average dwell time, and that the dwell time is exponentially distributed.

When motions result in angle changes of discrete size, the Kolmogorov equations (8–11) determine the probability of moving from X_1 to X_2. The variable X can always be made a discrete integer by suitably choosing the units on the abscissa of Fig. 2. Although not essential, it helps to visualize the process if such a transformation is assumed to have been made. The Kolmogorov equations pertain to such discrete (integer) steps when the dwell time is exponentially distributed.

The probability P_{ij} of finding the system in state j at time $\tau + T$ if at time τ it was in state i is needed. The probability that the system goes from state i to j in one step is denoted by p_{ij}. The values of p_{ij} can be determined from Fig. 2 as follows: except near the boundaries, where truncation might occur, p_{ij} depends only upon the difference $\Delta X = (j - i)$ because the steps are assumed to be independent from each other.

If the system is initially in state i, the probability that the system is in state k after n movements is

$$P_{ik}{}^{(n)} = \sum_j p_{ij} P_{jk}{}^{(n-1)} \quad (13)$$

where the system moves to an intermediate state j upon leaving i. Additionally, the probability that one or more motions have occurred in time t is equal to the probability that the waiting time until the first motion occurs is less than t, or, from Eq. (11)

$$P[n \neq 0] = 1 - \exp(-\lambda t) = P[T \leq t] = \int_0^t \epsilon(\tau) d\tau$$

from which, by Liebnitz rule, the probability density function for the waiting time in any state i is

$$\epsilon_i(t) = \lambda_i e^{-\lambda_i t} \quad (14)$$

The probability that the system moves from states i to k in time T is found by combining Eqs. (13) and (14). First, picture the system remaining in state i for some period [with probability density function (14)], then jumping to an intermediate state j (with probability p_{ij}), then progressing (along some path) from j to k in time $T - \tau$ (with probability P_{jk}). This motion is convoluted over all values of τ, and summed over all possible intermediate states j. Thus

$$P_{ik} = \delta_{ik} e^{-\lambda_i T} + \sum_j \int_0^T \lambda_i e^{-\lambda_i \tau} p_{ij} P_{jk}(T - \tau) d\tau \quad (15)$$

where the first term admits that the system may remain in state i throughout the period T. Equation (15) is the backward Kolmogorov equation. The forward Kolmogorov equation is obtained by first expressing the last jump explicitly

$$P_{ik}{}^{(n)} = \sum_j P_{ij}{}^{(n-1)} p_{jk} \quad (16)$$

from which

$$P_{ik} = \delta_{ik} e^{-\lambda_i T} + \int_0^T \sum_j P_{ij}(\tau) p_{jk} \lambda_j e^{-\lambda_k (T-\tau)} d\tau \quad (17)$$

There is a continuum analog to either of the aforementioned discrete step Kolmogorov equations. Let $f(X_1,t_1;X_2,t_2)$ be the (conditional) probability density function for angular deviation from X_1 to X_2 during the period t_1 to t_2. If this function is independent of the history prior to t_1

$$f(X_1,t_1;X_3,t_3) = \int_{-\infty}^{\infty} f(X_1,t_1;X_2,t_2)\, f(X_2,t_2;X_3,t_3)\, dX_2 \quad (18)$$

where the integral limits include all possible intermediate states, X_2. From Eq. (18), a continuous analog of the backward Kolmogorov equation is

$$-\partial f(X_1,t_1;X_3,t_3)/\partial t_1 = b(X_1,t_1)\partial f(X_1,t_1;X_3,t_3)/\partial X_1 + (\tfrac{1}{2})a(X_1,t_1)\partial^2 f(X_1,t_1;X_3,t_3)/\partial X_1{}^2 \quad (19)$$

where $b(X_1,t_1)$ and $a(X_1,t_1)$ are the infinitesimal means and variances for changes in X, defined by

$$b(X_1,t_1) = \lim_{\Delta t\to 0}(1/\Delta t) \int_{-\infty}^{\infty} (X_2 - X_1) f(X_1,t_1 - \Delta t;X_2,t_2) dX_2 \quad (20)$$

$$a(X_1,t_1) = \lim_{\Delta t\to 0}(1/\Delta t) \int_{-\infty}^{\infty} (X_2 - X_1)^2 f(X_1,t_1 - \Delta t;X_2,t_2) dX_2 \quad (21)$$

In the present problem, the transition probabilities depend only upon the time elapse, $t_3 - t_1$, and upon the difference in state, $X_3 - X_1$; therefore,

$$f(X_1,t_1;X_3,t_3) = u(X_3 - X_1;t_3 - t_1) \quad (22)$$

where u is the probability density function for the angular displacement and is homogeneous in time and additive in spatial coordinates. From Eq. (20), the mean change in position during Δt equals the product of the mean number of steps, $\lambda\Delta t$, and the mean step size $E(\Delta X)$. Thus

$$b(X,t) = \lambda E(\Delta X) = b \quad (23)$$

Similarly

$$a(X,t) = \lambda \operatorname{Var}(\Delta X) = a \quad (24)$$

Both $E(\Delta X)$ and $\operatorname{Var}(\Delta X)$ are obtained from Fig. 2. For the present problem a and b are constants.

If the forms given in Eqs. (22, 23, and 24) are substituted into Eq. (19), there results

$$\partial u/\partial t = (a/2)(\partial^2 u/\partial X^2) - b\partial u/\partial X \quad (25)$$

Equation (25) is applicable to the present random walk problem when a large number of steps is likely to be used to reach a boundary.

Solution to the Stochastic Equation

The "waiting time" for the spacecraft to rotate from its nominal orientation to some preselected limiting boundary is the inverse of the frequency of control jet firing. To find the waiting time, we must first find the probability that the spacecraft angular deviation is between X_1 and X_2, which, at t, is

$$P[X_1 < X < X_2,t] = \int_{X_1}^{X_2} u(y,t) dy \quad (26)$$

where $u(X,t)$ is first determined by solving Eq. (25).

Let the nominal (starting) position be $X_0 > 0$, with boundaries at $X = 0$ and $X = X_b > X_0$. The initial condition, in probabilistic terms, is that at $t = 0$, $X = X_0$ with probability one, or

$$u(X,0) = \delta(X - X_0) \quad (27)$$

where $\delta(X - X_0)$ is the Dirac delta function.

The use of "absorbing" boundaries facilitates finding the excursion time to reach a boundary. Define a "boundary state" as the state of the spacecraft immediately after the control jets fire; the jets fire when $X = X_b$ or $X = 0$. Thus,

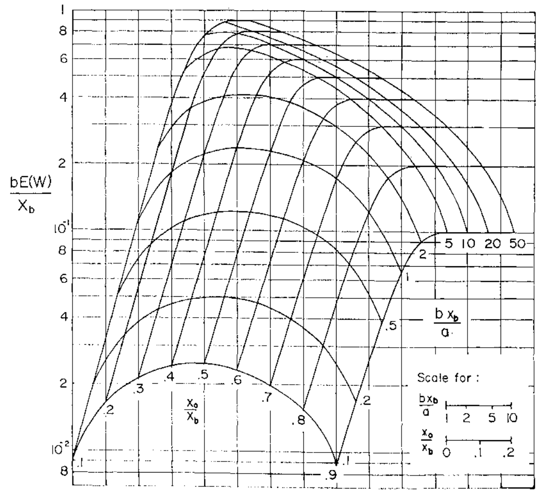

Fig. 3 Plot of the mean waiting time as a function of the starting position and the distance between the boundaries. The data are from the solution to the diffusion equations, Eq. (42) in the text. The lines are marked with the values of the parameter which is held constant.

the probability is zero that $X = X_b + \Delta X$ or $X = 0 - \Delta X$, or

$$0 = \int_{-\Delta X}^{0} u(y,t)\, dy = \int_{X_b}^{X_b+\Delta X} u(y,t)\, dy$$

and thus

$$u(X_b,t) = u(0,t) = 0 \qquad (28)$$

Equation (25), with conditions (27) and (28), is solved by iterated Laplace transforms. If $f(X,s)$ is the transform of $u(X,t)$, and $g(p,s)$ is the transform of $f(X,s)$, then the equations become

$$(-2\, sg/a) + p^2 g - f'(0,s) - (2\, bpg/a) = (-2/a) \exp(-pX_0) \qquad (29)$$

where $f'(0,s)$ is the derivative of $f(X,s)$ with respect to X evaluated at $X = 0$.

Solving Eq. (29) for g and taking the inverse transform gives

$$f_2(X,s) = 2 \exp(bX/a - bX_0/a) \sinh lX_0 \sinh(X_b - lX)/(la \sinh lX_b) \text{ for } X \geq X_0 \qquad (30a)$$

$$f_1(X,s) = 2 \exp(bX/a - bX_0/a) \sinh(lX_b - lX_0) \sinh lX/(la \sinh lX_0) \text{ for } X \geq X_0 \qquad (30b)$$

in which

$$l = (b/a)\,(1 + 2as/b^2)^{1/2} \qquad (31)$$

The probability that a boundary has not been reached at time t is

$$P[0 < X < X_b, t] = \int_0^{X_b} u(y,t) dy \qquad (32)$$

Let W be the time required to reach a boundary starting from X_0 at $t = 0$. By definition, W is the waiting time. The probability that $W > t$ is equal to the probability that a boundary has not reached at time t or

$$P[W > t] = \int_0^{X_b} u(y,t)\, dy \qquad (33)$$

To find the mean waiting time, the probability density function for W must be obtained from Eq. (33). Let $\beta(t)$ be this probability density function; then

$$\int_t^{\infty} \beta(\tau) d\tau = \int_0^{X_b} u(y,t) dy \qquad (34)$$

or

$$\beta(t) = -(\partial/\partial t) \int_0^{X_b} u(y,t) dy \qquad (35)$$

The mean value of the waiting time is

$$E(W) = \int_0^{\infty} t\beta(t) dt \qquad (36)$$

The variance is

$$\mathrm{Var}(W) = \int_0^{\infty} t^2 \beta(t) dt - E^2(W) \qquad (37)$$

The properties of the Laplace transform allow the operations indicated in Eqs. (35–37) to be performed with the transformed variables. Integrating Eq. (30) from 0 to X_b yields

$$F(s) = 2\{1 - \exp(-2lX_b) - [\exp(-lX_0) - \exp(-2lX_b + 2lX_0)] \exp(-bX_0/a) - [\exp(-lX_b + lX_0) - \exp(-lX_b - lX_0)] \exp(bX_b/a - bX_0/a)\}/\{a[1 - \exp(-2lX_b)](l^2 - b^2/a^2)\} \qquad (38)$$

By using the following Laplace transform operations

$$L\{(d/dt)L^{-1}[F(s)]\} = sF(s) - F(0) \qquad (39)$$

$$L[tF(t)] = [dF(s)/ds] \qquad (40)$$

$$\lim_{s\to 0} [-(d/ds)L(\beta)] = \lim_{s\to 0} \int_0^{\infty} t\beta(t) \exp(-st) dt = E(W) \qquad (41)$$

the mean waiting time and variance can be calculated.

$$E(W) = (X_b/b)\{[1 - \exp(-2bX_0/a)]/[1 - \exp(-2bX_b/a)] - (X_0/X_b)\} \qquad (42)$$

$$\mathrm{Var}(W) = \{(X_b/b)^2/[1 - \exp(-2bX_b/a)]^2\}\{[1 - \exp(-2bX_0/a)] \exp(-2bX_0/a) - 4(X_0/X_b)\,[1 - \exp(-2bX_b/a)] \exp(-2bX_0/a) + 3[1 - \exp(-2bX_0/a)] \exp(-2bX_b/a)\} + (a/b^2)E(W) \qquad (43)$$

Equations (42) and (43), along with (23) and (24) and a histogram, such as Fig. 2, constitute the solution to the problem. $E(W)$ and $\mathrm{Var}(W)$ are plotted in Figs. 3 and 4.

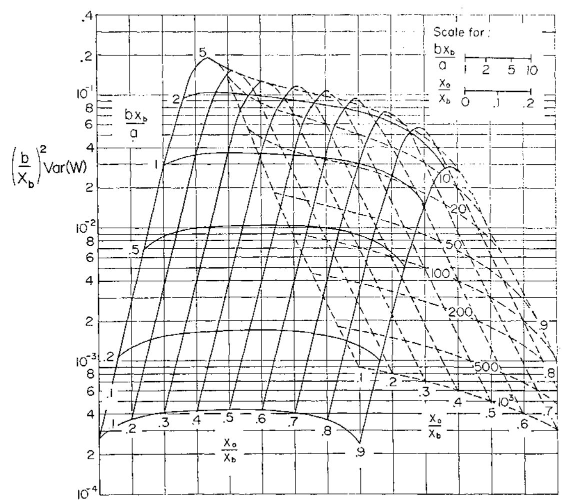

Fig. 4 Plot of the variance for the waiting time as a function of the starting position and distance between the boundaries. The data are from the solution to the diffusion equation, Eq. (43) in the text. The lines are marked with the values of the parameter which is held constant.

If the mean rate of motion b is not zero, the rate of control fuel consumption may be reduced by selecting the nominal position X_0 so that the waiting time is maximized. Equation (42) is differentiated with respect to X_0 and the result set equal to zero. This yields

$$(X_0/X_b)_{\text{opt}} = (-a/2bx_b)\ln(a/2bx_b) - 1 \qquad (44)$$

The previous solution is valid when many small steps are taken to reach a boundary, so that a continuum is approximated. However, when few steps are needed the Kolmogorov Eqs., (15) or (17), must be used. An iterative solution to Eqs. (15) and (17) will be presented next.

For the discrete-step analysis, let $\tilde{p}$ represent the transition matrix, whose elements are the one-step probabilities p_{ik}. Except when a step would carry the spacecraft beyond a boundary,

$$p_{ik} = P[\Delta X = (k - i)]$$

where ΔX is found from a histogram similar to Fig. 2. If the step would carry the spacecraft beyond the boundary, the probabilities are adjusted so that the spacecraft angular deviation X falls into a boundary state, labeled either "O" and located just outside the lower limiting state or labeled "γ" and just above the upper limit state (see the left and right columns of Fig. 5). When the next motion occurs, the spacecraft is reflected, with probability one, from a boundary state to the nominal position. When the two fictitious boundary states are added to $\tilde{p}$, a new matrix $\hat{p}$ is defined.

The dwell time is, as before, $1/\lambda$ for all states inside the physical boundaries. For the fictitious boundary states, the dwell time is taken as $1/\lambda_0$ to distinguish it for future manipulation.

A theorem—which can be found in Feller[8]—is useful: Suppose that, for a family of stochastic matrices $\tilde{P}(t)$ which have time dependent elements,

$$\tilde{P}(t + \tau) = \tilde{P}(t)\tilde{P}(\tau) \qquad (45)$$

and

$$\tilde{P}(t) \to \tilde{I} \text{ as } t \to 0$$

then, if no P_{ik} vanishes identically,

$$\lim_{t\to\infty} P_{ik}(t) = \nu_k \qquad (46)$$

where either $\nu_k = 0$ for all k or else

$$\nu_k > 0, \sum_k \nu_k = 1 \qquad (47)$$

and

$$\sum_j \nu_j P_{jk}(\tau) = \nu_k \qquad (48)$$

The second alternative occurs whenever there exists a probability vector ν_k satisfying Eq. (48) for some $\tau > 0$. In this case Eq. (48) holds for all $\tau > 0$ and the probability vector ν_k is unique and is the reciprocal of the mean recurrence time of state k.

To illustrate, suppose the spacecraft started at an initial angle $X_i = i$. After a long period during which the jets have fired many times, the probability that $X_j = j$ is essentially independent of the initial position. According to the above ergodic theorem, this probability is ν_j.

The Laplace transform of the forward Kolmogorov Eq. (17) is, in terms of the modified transition matrix (see Feller,[8] pp. 457–469),

$$\pi_{ik}(s) = [\delta_{ik}/(s + \lambda_i)] + \sum_{j=0}^{b} [\lambda_j/(s + \lambda_k)]\pi_{ij}(s)\hat{p}_{jk} \qquad (49)$$

where π_{ik} is the Laplace transform of P_{ik}. This equation is more conveniently manipulated in matrix form. Define: a

$\frac{1}{32}$ ×

																					State	
0	0	0	0	0	0	0	0	32	0	0	0	0	0	0	0	0	0	0	0	0	0	
13	5	3	4	1	2	1	0	2	1	0	0	0	0	0	0	0	0	0	0	0	1	
9	4	5	3	4	1	2	1	0	2	1	0	0	0	0	0	0	0	0	0	0	2	
6	3	4	5	3	4	1	2	1	0	2	1	0	0	0	0	0	0	0	0	0	3	
3	3	3	4	5	3	4	1	2	1	0	2	1	0	0	0	0	0	0	0	0	4	
1	2	3	3	4	5	3	4	1	2	1	0	2	1	0	0	0	0	0	0	0	5	
0	1	2	3	3	4	5	3	4	1	2	1	0	2	1	0	0	0	0	0	0	6	
0	0	1	2	3	3	4	5	3	4	1	2	1	0	2	1	0	0	0	0	0	7	
0	0	0	1	2	3	3	4	5	3	4	1	2	1	0	2	1	0	0	0	0	8	Nominal
0	0	0	0	1	2	3	3	4	5	3	4	1	2	1	0	2	1	0	0	0	9	
0	0	0	0	0	1	2	3	3	4	5	3	4	1	2	1	0	2	1	0	0	10	
0	0	0	0	0	0	1	2	3	3	4	5	3	4	1	2	1	0	2	1	0	11	
0	0	0	0	0	0	0	1	2	3	3	4	5	3	4	1	2	1	0	2	1	12	
0	0	0	0	0	0	0	0	1	2	3	3	4	5	3	4	1	2	1	0	3	13	
0	0	0	0	0	0	0	0	0	1	2	3	3	4	5	3	4	1	2	1	3	14	
0	0	0	0	0	0	0	0	0	0	1	2	3	3	4	5	3	4	1	2	4	15	
0	0	0	0	0	0	0	0	0	0	0	1	2	3	3	4	5	3	4	1	6	16	
0	0	0	0	0	0	0	0	0	0	0	0	1	2	3	3	4	5	3	4	7	17	
0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	3	3	4	5	3	11	18	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	3	3	4	5	14	19	
0	0	0	0	0	0	0	0	32	0	0	0	0	0	0	0	0	0	0	0	0	γ	

Fig. 5 The augmented matrix $\hat{p}$. The dashed lines correspond to the physical boundaries. The fictitious boundary states at 0 and γ are reflecting boundaries which reflect the spacecraft to the nominal position (chosen here as state 8) with probability one.

square diagonal matrix $\tilde{s}$ whose diagonal elements equal s; a square diagonal matrix $\tilde{\lambda}$ whose diagonal elements are equal to λ (except for elements $\lambda_{00} = \lambda_{\gamma\gamma} = \lambda_0$); and $\tilde{I}$ as the identity matrix. The summation runs from 0 to γ for the limited motion being considered. In terms of these matrices, Eq. (49) is

$$\pi(\tilde{s} + \tilde{\lambda}) = \tilde{I} + \pi\tilde{\lambda}\hat{p} \qquad (50)$$

For very long times, the ergodic theorem states that

$$\nu\hat{P} = \nu \qquad (51)$$

where, as introduced before, ν is a row matrix. Taking the Laplace transform of Eq. (51), leads to

$$\nu\tilde{s}\pi = \nu \qquad (52)$$

Substituting Eq. (52) into Eq. (50) yields

$$\nu\tilde{\lambda} = \nu\tilde{\lambda}\hat{p} \qquad (53)$$

Define a new row matrix η such that

$$\eta = \nu\tilde{\lambda} \qquad (54)$$

Equation (53), in terms of the row matrix η, is

$$\eta = \eta\hat{p} \qquad (55)$$

When the η_i are found, the ν_i are determined from Eq. (54). The ratios of the η_i with respect to one value, say η_N, can be determined by iteration from Eq. (55). Because both $\hat{P}$ and ν are stochastic matrices

$$\sum_j P_{ij} = \sum_j \nu_j = 1 \qquad (56)$$

which is the one additional equation needed to establish specific values for the ν_i.

The solution procedure is as follows: $\hat{p}$ is known; arbitrary initial values for η_i are chosen and new η_i are calculated using Eq. (55); the new matrix η is again multiplied by $\hat{p}$, and this iteration process is continued until no change (within the desired accuracy) occurs in the ratios of the η_i. The long-time probabilities are then found from Eqs. (54) and (56). Thus

$$\nu = [(\eta_0/\eta_N\lambda_0), (\eta_1/\eta_N\lambda), \ldots, (\eta_\gamma/\eta_N\lambda_\gamma)] \qquad (57)$$

and

$$[(\eta_0 + \eta_\gamma)/\lambda_0] + (1/\lambda)\sum_{i=1}^{\gamma-1} \eta_i = \eta_N \qquad (58)$$

The ergodic theorem states, finally, that the inverse of the probability of finding the system in state j at time $t \to \infty$ is proportional to the mean period between arrivals at state j. Define an "off period" as the period during which the space-

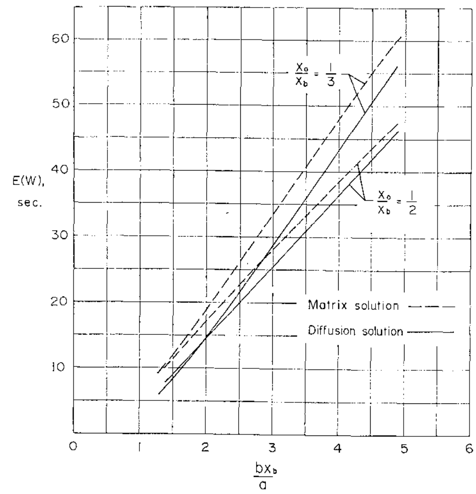

Fig. 6 Comparison of the expected waiting times for the matrix solutions and the diffusion equation solution. For this comparison $b = 1.0625$, $a = 21.124$, λ was taken as two movements per second, and the distribution of Fig. 2 was used.

craft is not at the boundary state $X_\gamma = \gamma$; a cycle starts when the spacecraft enters state X_γ and ends when it next reaches this state. Define the average length of an off period to be ζ_γ. The average length of time spent in state γ is $(1/\lambda_\gamma)$. The ratio of the time spent in state γ to the time spent in all states is the probability that the system is in state γ. That is,

$$\nu_\gamma = [1/(1 + \zeta\lambda_\gamma)] \tag{59}$$

The total time spent between entries to state γ, T_γ, can be put into a convenient form by using Eq. (58).

$$T_\gamma = (1/\lambda_\gamma) + \zeta_\gamma = (1/\eta_\gamma)\left\{[(\eta_0 + \eta_\gamma)/\lambda_0] + (1/\lambda)\sum_{i=1}^{\gamma-1} \eta_i\right\} \tag{60}$$

So that the cycle will not be prolonged by the introduction of the fictitious boundary states, the dwell time in states X_0 and X_γ can be made zero by letting $\lambda_0 \rightarrow \infty$.

$$\lim_{\lambda_0 \rightarrow \infty} T_\gamma = \zeta = E(W_\gamma) = [1/(\lambda\eta_\gamma)] \sum_{i=1}^{\gamma-1} \eta_i \tag{61}$$

where $E(W_\gamma)$ is the mean waiting time to reach the boundary X_γ, if the spacecraft starts from the nominal position.

Similarly, the mean waiting time to reach the boundary X_0 is

$$E(W_0) = [1/(\lambda\eta_0)] \sum_{i=1}^{\gamma-1} \eta_i \tag{62}$$

The mean waiting time to reach either boundary is

$$E(W) = \{1/[\lambda(\eta_0 + \eta_\gamma)]\} \sum_{i=1}^{\gamma=1} \eta_i \tag{63}$$

Equations (55, 60, 61, and 63) along with a modified matrix $\hat{p}$ constitute the solution for the various waiting times.

Numerical Results and Discussion

The mean value b/λ and variance a/λ were determined from a histogram like Fig. 2. For representative numerical calculations, the number of moves per sec, λ, was arbitrarily chosen at two moves per sec. However, since the results are linearly dependent on λ, they can be scaled to any other rate.

The maximum square matrix size for the iteration method was limited by computer storage to 100×100 which, with the histogram of Fig. 2, corresponds to a case where the allowed maximum deviation is very small (tight boundaries). The iteration scheme converged with a 50×50 matrix in less than 200 iterations with a relative error of less than 10^{-5} for all elements in the matrix. The computing time was less than 7 secs on a CDC 6600 computer.

The expected waiting time, as determined by the solution to the diffusion equation, is shown in Fig. 3. The waiting time essentially reduces to the deterministic solution for values of $(bX_b/a > 10)$. [The deterministic solution is $E(W) = (X_b - X_0)$, $b > 0$.] The figure also illustrates that the maximum mean waiting time occurs at smaller values of X_0/X_b as bX_b/a increases.

Figure 4 illustrates the nondimensional variance. The nondimensional variance reaches a maximum for $2.4 < bX_b/a < 5.3$ for $0.1 < X_0/X_b < 0.9$, or approximately when the waiting time reduces to the deterministic form.

The diffusion equation solution is compared to the matrix solution in Fig. 6. The matrix sizes used for the calculation ranged from 40×40 to 99×99, where the fictitious states are included in the matrix size. The values $b/\lambda = 0.53125$, $a/\lambda = 10.562$, and $\lambda = 2$ were used. The agreement with the iteration method is good for $X_0/X_b = \frac{1}{2}$ and improves at the larger values of bX_b/a. The largest value of bX_b/a shown for $X_0/X_b = \frac{1}{2}$ corresponds to a 99×99 matrix, for which an average of 92 steps are required to reach a boundary. The difference is approximately 3% at this point and decreasing absolutely and relatively with increasing bX_b/a because, as the number of steps increase, the conditions for the diffusion equation are more nearly approximated.

The case $X_0/X_b = \frac{1}{3}$ is also shown in Fig. 6, and the same general behavior between the curves is noted. However, the agreement is not as good as for $X_0/X_b = \frac{1}{2}$, even though at the upper end the mean number of steps required to reach a boundary is greater for $X_0/X_b = \frac{1}{3}$. This may be due to the character of the particular one-step matrix used.

For actual design calculations, the diffusion equation solution required less detailed information about the transition histogram; only the mean and variance need be known. On the other hand, for the matrix solution, a rather detailed histogram similar to Fig. 2 is needed.

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